

# BFKL predictions at small $x$ from $k_T$ and collinear factorization viewpoints

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## Abstract

Hard scattering processes involving hadrons at small  $x$  are described by a  $k_T$ -factorization formula driven by a BFKL gluon. We explore the equivalence of this description to a collinear-factorization approach in which the anomalous dimensions  $\gamma_{gg}$  and  $\gamma_{qg}/\alpha_S$  are expressed as power series in  $\alpha_S \log(1/x)$ , or to be precise  $\alpha_S/\omega$  where  $\omega$  is the moment index. In particular we confront the collinear-factorization expansion with that extracted from the BFKL approach with running coupling included.

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Recently there have been several studies [1-5] of the validity and possible modification of the conventional Altarelli-Parisi (or GLAP) description of deep inelastic scattering in the small  $x$  region that has become accessible at HERA,  $x \sim 10^{-4}$ . The relevant modifications are the inclusion of contributions which are enhanced by powers of  $\log(1/x)$ , but which lie outside the leading (and next-to-leading) Altarelli-Parisi perturbative expansion. Formally they correspond to the expansion of the anomalous dimensions  $\gamma_{gg}$  and  $\gamma_{qg}/\alpha_S$  as power series in  $\alpha_S/\omega$  where  $\omega$  is the moment index. An alternative approach which automatically resums *all* these leading  $\log(1/x)$  contributions to  $\gamma_{gg}$  and  $\gamma_{qg}/\alpha_S$  is provided by the BFKL equation coupled with the  $k_T$ -factorization formula for calculating observable quantities [6, 7]. The main aim of this paper is to explore the connection between these two approaches. To be specific we study the relation between the collinear-factorization formula with  $\log(1/x)$  terms included and the  $k_T$ -factorization formula based on the solution of the BFKL equation [8] with running coupling  $\alpha_S$ . We show that both approaches generate the same first few terms in the perturbative expansion of  $\gamma_{gg}$  and, more important, of  $\gamma_{qg}$ , which are presumably the most relevant contributions for the description of deep inelastic scattering in the HERA range. They differ substantially, however, in the asymptotically small  $x$  regime.

Deep inelastic unpolarised electron-proton scattering may be described in terms of two structure functions,  $F_2(x, Q^2)$  and  $F_L(x, Q^2)$ . As usual, the kinematic variables are defined to be  $Q^2 = -q^2$  and  $x = Q^2/2p \cdot q$ , where  $p$  and  $q$  are the four-momenta of the incoming proton and virtual photon probe respectively. At small values of  $x$ ,  $x \lesssim 10^{-3}$ , these observables reflect the distribution of gluons in the proton, which are by far the dominant partons in this kinematic region. The precise connection between the small  $x$  structure functions and the gluon distribution is given by the  $k_T$ -factorization formula [6, 7],

$$F_i(x, Q^2) = \int \frac{dk_T^2}{k_T^2} \int_x^1 \frac{dx'}{x'} F_i^{\gamma g}\left(\frac{x}{x'}, k_T^2, Q^2\right) f(x', k_T^2) \quad (1)$$

with  $i = 2, L$ , which is displayed pictorially in Fig. 1. The gluon distribution  $f(x, k_T^2)$ , unintegrated over  $k_T^2$ , is a solution of the BFKL equation, while  $F_i^{\gamma g}$  are the off-shell gluon structure functions which at lowest-order are determined by the quark box (and crossed-box) contributions to photon-gluon fusion, see Fig. 1.

For sufficiently large values of  $Q^2$  the leading-twist contribution is dominant, and it is most transparent to discuss the  $Q^2$  evolution of  $F_i(x, Q^2)$  in terms of moments. Then the  $x'$  convolution of (1) factorizes to give

$$\overline{F}_i(\omega, Q^2) = \int \frac{dk_T^2}{k_T^2} \overline{F}_i^{\gamma g}(\omega, k_T^2, Q^2) \overline{f}(\omega, k_T^2) \quad (2)$$

where the moment function

$$\overline{f}(\omega, k_T^2) \equiv \int_0^1 \frac{dx}{x} x^\omega f(x, k_T^2), \quad (3)$$

with similar relations for  $\overline{F}_i$  and  $\overline{F}_i^{\gamma g}$ .

## Fixed $\alpha_S$ : $k_T$ -factorization to collinear-factorization

It is illuminating to first consider the case of fixed coupling  $\alpha_S$  [2]. Then the photon-gluon moments,  $\overline{F}_i^{\gamma g}$  are simply functions of  $\tau \equiv Q^2/k_T^2$  (and  $\omega$ ), for massless quarks. Hence (2) becomes a convolution in  $k_T^2$  which, in analogy with the  $x'$  convolution, may be factorized by taking moments a second time. In this way we obtain representations for the  $\overline{F}_i$  with factorizable integrands

$$\overline{F}_i(\omega, Q^2) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} d\gamma \tilde{F}_i^{\gamma g}(\omega, \gamma) \tilde{f}(\omega, \gamma) (Q^2)^\gamma \quad (4)$$

with  $c = \frac{1}{2}$ . The (double) moments  $\tilde{F}_i^{\gamma g}$  and  $\tilde{f}$  of the gluon structure functions and the gluon distribution are respectively defined by

$$\tilde{F}_i^{\gamma g}(\omega, \gamma) = \int d\tau \tau^{-\gamma-1} \overline{F}_i^{\gamma g}(\omega, \tau) \quad (5)$$

$$\tilde{f}(\omega, \gamma) = \int dk_T^2 (k_T^2)^{-\gamma-1} \overline{f}(\omega, k_T^2) \quad (6)$$

where the  $\tilde{F}_i^{\gamma g}$  are dimensionless, but  $\tilde{f}$  carries the dimension  $(k_0^2)^{-\gamma-1}$ . Representation (4) enables the leading-twist contribution to be identified from a knowledge of the analytic properties of  $\tilde{f}$  and  $\tilde{F}_i^{\gamma g}$  in the complex  $\gamma$  plane.

The gluon distribution  $f(x, k_T^2)$  satisfies the BFKL equation, which in moment space has the form

$$\overline{f}(\omega, k_T^2) = f^0(\omega, k_T^2) + \frac{\overline{\alpha}_S}{\omega} \int \frac{dk_T'^2}{k_T'^2} K(k_T^2, k_T'^2) \overline{f}(\omega, k_T'^2) \quad (7)$$

where  $\overline{\alpha}_S \equiv 3\alpha_S/\pi$  and  $K$  is the usual BFKL kernel. The double-moment function  $\tilde{f}$  is therefore given by

$$\tilde{f}(\omega, \gamma) = \frac{\tilde{f}^0(\omega, \gamma)}{1 - (\overline{\alpha}_S/\omega) \tilde{K}(\gamma)} \quad (8)$$

where  $\tilde{K}(\gamma)$  is the eigenvalue of the BFKL kernel corresponding to the eigenfunction proportional to  $(k_T^2)^\gamma$ . It can be shown that

$$\begin{aligned} \tilde{K}(\gamma) &= 2\Psi(1) - \Psi(1-\gamma) - \Psi(\gamma) \\ &= \frac{1}{\gamma} \left[ 1 + \sum_{n=1}^{\infty} 2\zeta(2n+1) \gamma^{2n+1} \right] \end{aligned} \quad (9)$$

where  $\Psi$  is the logarithmic derivative of the Euler gamma function,  $\Psi(z) \equiv \Gamma'(z)/\Gamma(z)$ , and  $\zeta(n)$  is the Riemann zeta function.

We see from (4) that the large  $Q^2$  behaviour of  $\overline{F}_i(\omega, Q^2)$  is controlled by the pole at  $\gamma = \overline{\gamma}$  of  $\tilde{f}(\omega, \gamma)$  of (8) which lies to the left of, and nearest to, the contour of integration in the  $\gamma$ -plane. For a physically reasonable choice of input  $\tilde{f}^0$ , this pole arises from the zero of the denominator of (8). That is

$$\frac{1}{1 - (\overline{\alpha}_S/\omega) \tilde{K}(\gamma)} = \frac{\gamma R}{\gamma - \overline{\gamma}}, \quad (10)$$

where from (9) we have

$$\bar{\gamma} = \frac{\bar{\alpha}_S}{\omega} + 2\zeta(3) \left(\frac{\bar{\alpha}_S}{\omega}\right)^4 + 2\zeta(5) \left(\frac{\bar{\alpha}_S}{\omega}\right)^6 + \mathcal{O}\left(\frac{\bar{\alpha}_S}{\omega}\right)^7, \quad (11)$$

$$\begin{aligned} R &= \left(1 - \frac{\bar{\alpha}_S}{\omega} \frac{d(\gamma \tilde{K})}{d\gamma} \Big|_{\bar{\gamma}}\right)^{-1} \\ &= 1 + 6\zeta(3) \left(\frac{\bar{\alpha}_S}{\omega}\right)^3 + \mathcal{O}\left(\frac{\bar{\alpha}_S}{\omega}\right)^5. \end{aligned} \quad (12)$$

$\bar{\gamma}$  is the leading-twist anomalous dimension [9]. If we insert the pole of (10) and (8) into (4), and we close the contour of integration in the left-half plane, then we obtain the high  $Q^2$  behaviour

$$\bar{F}_i(\omega, Q^2) = \tilde{F}_i^{\gamma g}(\omega, \bar{\gamma}) \bar{\gamma} R \left(\frac{\bar{\alpha}_S}{\omega}\right) \tilde{f}^0(\omega, \bar{\gamma})(Q^2)^{\bar{\gamma}}. \quad (13)$$

Eq. (13) is the usual formula for the factorization of collinear (or mass) singularities written in moment space. This becomes more apparent if we express (13) in the form

$$\bar{F}_i(\omega, Q^2) = C_i^{\gamma g}(\omega, \bar{\gamma}) g(\omega, Q^2) \quad (14)$$

where

$$C_i^{\gamma g}(\omega, \bar{\gamma}) = \bar{\gamma} \tilde{F}_i^{\gamma g}(\omega, \bar{\gamma}) R \left(\frac{\bar{\alpha}_S}{\omega}\right) \quad (15)$$

is the moment of the (process dependent) coefficient function and

$$g(\omega, Q^2) = (Q_0^2)^{\bar{\gamma}} \tilde{f}^0(\omega, \bar{\gamma}) \left(\frac{Q^2}{Q_0^2}\right)^{\bar{\gamma}} \quad (16)$$

is the moment function of the (integrated) gluon density. Thus we can identify  $(Q_0^2)^{\bar{\gamma}} \tilde{f}^0(\omega, \bar{\gamma})$  with the moment of the gluon distribution at the “starting” scale  $Q_0^2$  of the evolution in  $Q^2$ .

The quantity  $R$ , the residue in (10), is renormalisation scheme dependent [2]. For studies of the BFKL equation, (7), it is appropriate to regularise  $R$  by choosing an inhomogeneous term of the form

$$f^0(\omega, k_T^2) = G^0(\omega) \delta(k_T^2 - \mu^2). \quad (17)$$

On the other hand  $\bar{\gamma}$ , and  $\tilde{F}_i^{\gamma g}$ , are scheme independent (at least in the so-called regular schemes). The scheme dependence of  $R$  is compensated by subleading contributions of  $\mathcal{O}(\alpha_S(\alpha_S/\omega)^n)$  in the anomalous dimension  $\gamma_{gg}$ , which are still at present unknown. This cancellation takes place when we allow the coupling to run.

The above collinear factorization formula (13) is true as it stands for  $F_L$ , but some care is needed for  $F_2$ . First we check its validity for  $F_L$ . Since  $\bar{F}_L^{\bar{\gamma}g} \rightarrow \text{constant}$  for large  $\tau$ , we see from (5) that  $\tilde{F}_L^{\gamma g}(\omega, \gamma) \sim 1/\gamma$ . However, this potential singularity at  $\gamma = 0$  is cancelled by the  $\gamma$  factor in the numerator of (10). On the other hand  $\bar{F}_2^{\gamma g} \rightarrow \log(\tau)$  and hence  $\tilde{F}_2^{\gamma g}(\omega, \gamma) \sim 1/\gamma^2$ ,

where the double pole reflects the collinear singularity associated with the  $g \rightarrow q\bar{q}$  transition. The integration contour in (4) therefore also encloses the pole

$$\tilde{F}_2^{\gamma g}(\omega, \gamma) \tilde{f}(\omega, \gamma) \sim \frac{1}{\gamma^2} \gamma \sim \frac{1}{\gamma}, \quad (18)$$

which gives rise to a “scaling sea” contribution to  $F_2$  which is independent of  $Q^2$ . To remove this contribution and to focus attention on the effects of the perturbative pole at  $\gamma = \bar{\gamma}$  we consider the observable  $\partial F_2 / \partial \log Q^2$ , rather than  $F_2$  itself. In this case the collinear factorization formula is of the form

$$\frac{\partial \bar{F}_2(\omega, Q^2)}{\partial \log Q^2} = \sum_q 2e_q^2 P_{qg}(\omega, \bar{\gamma}) g(\omega, Q^2) \quad (19)$$

where the coefficient, or gluon-quark splitting, function is given by

$$P_{qg}(\omega, \bar{\gamma}) = \bar{\alpha}_S \Phi(\omega, \bar{\gamma}) R\left(\frac{\bar{\alpha}_S}{\omega}\right), \quad (20)$$

with

$$\bar{\alpha}_S \Phi(\omega, \gamma) \equiv \gamma^2 \tilde{F}_2^{\gamma g}(\omega, \gamma) \quad (21)$$

defined to be a regular function at  $\gamma = 0$ . Since  $\Phi$  is known [2] in terms of the quark box (and crossed box), we can determine the perturbative expansion of  $P_{qg}$  by calculating

$$P_{qg}(\omega, \bar{\gamma}) = \bar{\alpha}_S \left[ \Phi(\omega, 0) + \bar{\gamma} \left. \frac{\partial \Phi}{\partial \gamma} \right|_{\gamma=0} + \dots \right] R \quad (22)$$

with  $\bar{\gamma}$  and  $R$  given by the perturbative expansions of (11) and (12) respectively. To be precise we substitute for  $\bar{\gamma}$  in (22) and obtain the perturbative expansion of  $P_{qg}/\alpha_S$  as a power series in  $\bar{\alpha}_S/\omega$ . This provides the recipe to compute the leading  $\log(1/x)$  contribution to  $P_{qg}$ .

Although we have expanded the observables in a perturbative series in  $\bar{\alpha}_S/\omega$ , we should recall that the small  $x$  behaviour of  $F_L$  and  $\partial F_2 / \partial \log Q^2$  is controlled by the singularities of  $\bar{F}_i(\omega, Q^2)$  in the  $\omega$  complex plane. The singularities arise from  $\bar{\gamma}$ . The leading singularity of  $\bar{\gamma}$  is the BFKL branch point at  $\omega = \omega_L = (4 \log 2) \bar{\alpha}_S$ . To see this we note that the position of the singularity is controlled by the value of  $\tilde{K}(\gamma)$  at its symmetry point,  $\gamma = \frac{1}{2}$ . We expand  $\tilde{K}$  about this point

$$\tilde{K}(\gamma) = 4 \log 2 + 14 \zeta(3) (\gamma - \frac{1}{2})^2 + \dots, \quad (23)$$

and determine the leading singularity as the implicit solution of

$$1 - \frac{\bar{\alpha}_S}{\omega} \tilde{K}(\bar{\gamma}) = 0, \quad (24)$$

see (8). The leading pole of  $\tilde{f}(\omega, \gamma)$ , which lies inside the contour of integration of (4), is at

$$\bar{\gamma} = \frac{1}{2} - \sqrt{\frac{\omega - \omega_L}{14 \bar{\alpha}_S \zeta(3)}} \quad (25)$$

where  $\omega_L = (4 \log 2) \bar{\alpha}_S$ . In  $x$  space the leading singularity of the anomalous dimension gives an  $x^{-\omega_L}$  behaviour of the gluon distribution at asymptotically small values of  $x$ . On the other hand the perturbation series in  $\bar{\alpha}_S/\omega$ , as in (11), enables the collinear factorization formula to be used to investigate the approach to the BFKL  $x^{-\omega_L}$  form, as  $x$  decreases since

$$\sum_{n=1} c_n \left( \frac{\bar{\alpha}_S}{\omega} \right)^n \rightarrow \sum_{n=1} c_n \bar{\alpha}_S \frac{(\bar{\alpha}_S \log 1/x)^{n-1}}{(n-1)!}. \quad (26)$$

The collinear factorization formulae give well-defined perturbative expansions for  $F_L$  and  $\partial F_2 / \partial \log Q^2$  which allow the leading  $\bar{\alpha}_S \log(1/x)$  contributions to be resummed. We will discuss the implications of the reduction of  $k_T$ -factorization to collinear form after we have implemented the running of  $\alpha_S$ .

## Running $\alpha_S$ : collinear-factorization to $k_T$ -factorization

To see the effect of the running of  $\bar{\alpha}_S$  we simply replace

$$\bar{\gamma}(\bar{\alpha}_S, \omega) \rightarrow \bar{\gamma}(\bar{\alpha}_S(Q^2), \omega), \quad (27)$$

and similarly for  $R(\bar{\alpha}_S/\omega)$ , in (14) and (19). The crucial change is in the  $Q^2$  evolution factor of  $g(\omega, Q^2)$ , which becomes

$$\left( \frac{Q^2}{Q_0^2} \right)^{\bar{\gamma}} \rightarrow \exp \left( \int_{Q_0^2}^{Q^2} \frac{dq^2}{q^2} \bar{\gamma}(\bar{\alpha}_S(q^2), \omega) \right). \quad (28)$$

In the small  $x$  BFKL limit  $\bar{\gamma}$  is simply a function of the ratio  $\bar{\alpha}_S(q^2)/\omega$ , as in the fixed coupling case. We see immediately the important role played by the non-perturbative region. To illustrate the effect, it is sufficient to take

$$\bar{\alpha}_S(q^2) = b / \log(q^2/\Lambda^2), \quad (29)$$

and to write (11) in the form

$$\bar{\gamma} = \sum_{n=1}^{\infty} A_n \left( \frac{\bar{\alpha}_S(q^2)}{\omega} \right)^n \quad (30)$$

where the coefficients are known (and in particular  $A_1 = 1$  and  $A_2 = A_3 = A_5 = 0$ ). Then the exponent in (28) is given by

$$\int_{Q_0^2}^{Q^2} \frac{dq^2}{q^2} \bar{\gamma} = \frac{b}{\omega} \log \left( \frac{\bar{\alpha}_S(Q_0^2)}{\bar{\alpha}_S(Q^2)} \right) + \frac{b}{\omega} \sum_{n=4}^{\infty} \frac{A_n}{n-1} \left\{ \left( \frac{\bar{\alpha}_S(Q_0^2)}{\omega} \right)^{n-1} - \left( \frac{\bar{\alpha}_S(Q^2)}{\omega} \right)^{n-1} \right\}. \quad (31)$$

The first term on the right-hand-side leads to the usual double-leading-logarithmic (DLL) behaviour of the gluon distribution  $g(\omega, Q^2)$ . The sum in the second term builds up the BFKL behaviour, and here we see the dominance of the  $\bar{\alpha}_S(Q_0^2)/\omega$  power series evaluated at the starting scale  $Q_0^2$  as compared to the truly perturbative power series in  $\bar{\alpha}_S(Q^2)/\omega$ . In other words the leading singularity in (31) is the BFKL branch point at  $\omega = \bar{\alpha}_S(Q_0^2) 4 \log 2$ . In

principle it should be reabsorbed in the starting distribution  $g(\omega, Q_0^2)$ , leaving the perturbative contribution which is controlled by  $\bar{\alpha}_S(Q^2)$ ; yet in practice it is the full formula (31) which is used [1, 3, 4, 5].

The above representation contains therefore an equivalent infrared sensitivity to that contained in the direct BFKL predictions [7], but we see that it has been explicitly isolated in a factorizable form. By infrared sensitivity we mean that the leading singularity in the  $\omega$  plane is controlled by  $\alpha_S(Q_0^2)$  and not by  $\alpha_S(Q^2)$ . The distinction between  $Q^2$  and  $Q_0^2$  is, of course, immaterial in the region ( $Q^2 \gtrsim Q_0^2$ ) of applicability of the genuine leading  $\log(1/x)$  approximation, that is  $\alpha_S(Q_0^2) \log(Q^2/Q_0^2) \ll 1$ , but  $\alpha_S(Q_0^2) \log(1/x) \sim \mathcal{O}(1)$ .

The BFKL corrections to the DLL contribution only enter the expansion for the anomalous dimension (31) at order  $(\alpha_S/\omega)^4$  and above, whereas for  $P_{gg}$  of (23) it can be shown that all terms ( $n = 0, 1, \dots$ ) are present in the expansion  $\sum B_n(\alpha_S/\omega)^n$ . For this reason we expect that the small  $x$  behaviour of  $\partial F_2/\partial \log Q^2$  in the HERA regime will be controlled more by the perturbative expansion of  $P_{gg}$  than of  $\bar{\gamma}$ . However, as  $x$  decreases the expansion of  $\bar{\gamma}$  will begin to play a dominant role.

The BFKL equation was originally derived for fixed  $\alpha_S$ . The correct way to include the running of  $\alpha_S$  is not firmly established. The procedure usually adopted is to take  $\alpha_S(k_T^2)$  in (7) so that the DLL limit of GLAP evolution is obtained. Here we find the perturbative expansion obtained from this prescription. We are therefore able to check the validity of the procedure by comparing with the expansion obtained from the renormalization group (or collinear factorization) approach, that is (27)-(31).

If we replace the fixed  $\alpha_S$  of (7) by  $\alpha_S(k_T^2)$  the BFKL equation becomes

$$\log\left(\frac{k_T^2}{\Lambda^2}\right) \bar{f}(\omega, k_T^2) = \log\left(\frac{k_T^2}{\Lambda^2}\right) f^0(\omega, k_T^2) + \frac{b}{\omega} \int \frac{dk_T'^2}{k_T'^2} K(k_T^2, k_T'^2) \bar{f}(\omega, k_T'^2), \quad (32)$$

which, in terms of the moment variable  $\gamma$  conjugate<sup>2</sup> to  $k_T^2/\Lambda^2$ , reduces to the differential equation [10-13]

$$-\frac{\partial \tilde{f}(\omega, \gamma)}{\partial \gamma} = -\frac{\partial \tilde{f}^0(\omega, \gamma)}{\partial \gamma} + \frac{b}{\omega} \tilde{K}(\gamma) \tilde{f}(\omega, \gamma). \quad (33)$$

From the extension of (4) and (21) to running  $\alpha_S$  we see that the  $k_T$ -factorization gives

$$\frac{\partial F_2(\omega, Q^2)}{\partial \log Q^2} = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} d\gamma \bar{\alpha}_S(Q^2) \Phi(\omega, \gamma) \frac{1}{\gamma} \tilde{f}(\omega, \gamma) \left(\frac{Q^2}{\Lambda^2}\right)^\gamma, \quad (34)$$

and similarly for  $F_L(\omega, Q^2)$ , where the double-moment of the gluon  $\tilde{f}(\omega, \gamma)$  is the solution of (33). The leading-twist contribution is controlled by the solution of the homogeneous form of (33) [11, 12]

$$\tilde{f}(\omega, \gamma) = H^0(\omega) \exp \left[ \frac{b}{\omega} \int_\gamma d\gamma' \tilde{K}(\gamma') \right] \quad (35)$$

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<sup>2</sup>For running  $\alpha_S$  we choose to inter-relate dimensionless quantities  $\bar{f} \leftrightarrow \tilde{f}$ , whereas for fixed  $\alpha_S$  it was convenient to allow  $\tilde{f}$  to carry dimensions, see (6).

where  $H^0(\omega)$  will eventually have to be fixed by the starting gluon distribution,  $\tilde{f}^0$ .

We now inspect the perturbative expansion of (34) and find that the first few terms are identical to those in the renormalization group expansion. To make this identification we concentrate on the expansion in terms of powers of  $\bar{\alpha}_S(Q^2)/\omega$  which contain the hard scale  $Q^2$ . The non-perturbative contributions can always be absorbed into a redefinition of  $H^0(\omega)$ .

To begin we note that the leading twist contribution is controlled by the strip  $-1 < \gamma < 0$  of the branch cut in the  $\gamma$  plane, where the branch point at  $\gamma = 0$  is generated entirely by the  $1/\gamma$  term in  $\tilde{K}(\gamma)$ . We isolate this singularity by introducing the function

$$\overline{K}(\gamma) \equiv \tilde{K}(\gamma) - \frac{1}{\gamma} \quad (36)$$

such that  $\overline{K}$  is regular at  $\gamma = 0$ . We then insert the result

$$\frac{b}{\omega} \int_{\gamma} d\gamma' \tilde{K}(\gamma') = -\frac{b}{\omega} \log \gamma + \frac{b}{\omega} \int_{\gamma} d\gamma' \overline{K}(\gamma') \quad (37)$$

into (35) and (34), fold the contour around the cut and evaluate the discontinuity to obtain

$$\frac{\partial F_2(\omega, Q^2)}{\partial \log Q^2} = -\sin\left(\frac{\pi b}{\omega}\right) H^0(\omega) \bar{\alpha}_S(Q^2) I + \text{higher twist} \quad (38)$$

where

$$I \equiv \int_{-1}^0 d\gamma \Phi(\omega, \gamma) (-\gamma)^{-\frac{b}{\omega}-1} \exp\left(\frac{b}{\omega} \int_{\gamma} d\gamma' \overline{K}(\gamma')\right) \left(\frac{Q^2}{\Lambda^2}\right)^{\gamma}. \quad (39)$$

The integral  $I$  has, of course, to be understood in the sense of an analytic continuation since it diverges at  $\gamma = 0$ . To expand  $I$  in a perturbation series we first change the variable of integration

$$\gamma \rightarrow -\rho/t \quad \text{where} \quad t \equiv \log(Q^2/\Lambda^2), \quad (40)$$

then the integral takes the form

$$I = t^{b/\omega} \int_0^t d\rho \Phi(\omega, -\rho/t) \rho^{-\frac{b}{\omega}-1} \exp\left(\frac{b}{\omega} \int_{-\rho/t} d\gamma' \overline{K}(\gamma')\right) e^{-\rho}. \quad (41)$$

We use (9) to expand the first exponential factor in (41)

$$\exp\left(\frac{b}{\omega} \int_{-\rho/t} d\gamma' \overline{K}(\gamma')\right) = \exp\left(\frac{b}{\omega} \sum_{n=1} \frac{2\zeta(2n+1)}{2n+1} \left(\frac{\rho}{t}\right)^{2n+1}\right), \quad (42)$$

where we have omitted a factorizable non-perturbative contribution coming from the upper limit. When we expand the exponential in (42) and insert the series into (41) we encounter integrals of the form

$$\begin{aligned} \int_0^\infty d\rho \rho^{-\frac{b}{\omega}+2n} e^{-\rho} &= \Gamma\left(-\frac{b}{\omega} + 2n + 1\right) \\ &= \Gamma(-b/\omega) (-b/\omega)^{2n+1} (1 + \mathcal{O}(\omega)). \end{aligned} \quad (43)$$



Here the contribution from  $t < \rho < \infty$  gives higher-twist terms which vanish as  $1/Q^2$ , modulo logarithmic corrections. The term  $\Gamma(-b/\omega)$  can be reabsorbed into the starting distribution, where it belongs, and we find the perturbative expansion of  $I$  is of the form

$$I \sim t^{b/\omega} \Gamma(-b/\omega) \Phi(\omega, 0) \left[ 1 - \frac{b}{\omega} \sum_{n=1} \frac{2\zeta(2n+1)}{2n+1} \left( \frac{\bar{\alpha}_S(Q^2)}{\omega} \right)^{2n+1} \right] + \text{higher order terms.} \quad (44)$$

We see that the first two terms ( $n = 1, 2$ ) are identical to the first two terms ( $n = 4, 6$ ) in the perturbative expansion of (31), which are proportional to  $\bar{\alpha}_S(Q^2)^3$  and  $\bar{\alpha}_S(Q^2)^5$  respectively. The DLL contribution in (31) corresponds to the  $t^{b/\omega}$  factor in (44).

To generate the expansion of  $P_{gg}/\alpha_S$  as a power series in  $\bar{\alpha}_S/\omega$  we expand the function  $\Phi(\omega, \gamma)$  of (41) around  $\gamma = 0$ . This procedure generates the same first three terms as those in the expansion shown in (22). At higher order,  $(\bar{\alpha}_S/\omega)^3$  and above, we see that the terms of  $\mathcal{O}(\omega)$  in (43), as well as various other contributions, will also contribute to the expansion of  $P_{gg}$ . However, it is the first few terms that are important for the onset of the BFKL behaviour in the HERA small  $x$  regime [1, 3]. Note that the perturbative expansion in (44) contains an additional factor of  $b/\omega$  which enables this series to be separated from the perturbative expansion of  $\Phi(\omega, \gamma)$ .

Before we conclude, we can gain further insight into the relation between the BFKL equation and collinear factorization in the case of running  $\alpha_S$  if we estimate the integral (34) using the saddle-point method. Inserting (35) we see that the position of the saddle-point,  $\bar{\gamma}$ , is given by the implicit equation

$$-\frac{b}{\omega} \tilde{K}(\bar{\gamma}) + \log \left( \frac{Q^2}{\Lambda^2} \right) = 0,$$

that is by

$$\frac{\bar{\alpha}_S(Q^2)}{\omega} \tilde{K}(\bar{\gamma}) = 1, \quad (45)$$

which is the same as (24) for fixed  $\alpha_S$ . We evaluate the integrand of (34) at  $\gamma = \bar{\gamma}$  and use (45) to rearrange the product  $\tilde{f}(\omega, \bar{\gamma}) (Q^2/\Lambda^2)^{\bar{\gamma}}$  in the form

$$H^0(\omega) \exp \left\{ \left( \frac{b}{\omega} \int_{\bar{\gamma}} d\gamma' \tilde{K}(\gamma') \right) + \bar{\gamma} \log \left( \frac{Q^2}{\Lambda^2} \right) \right\} = \hat{H}^0(\omega) \exp \left\{ \int_{Q_0^2}^{Q^2} \frac{dq^2}{q^2} \bar{\gamma}(\bar{\alpha}_S(q^2)/\omega) \right\} \quad (46)$$

where  $\hat{H}^0$  includes the integration constant. The equality (46) is obtained by integrating the integral on the left-hand-side by parts. Thus the saddle-point estimate of (34) gives

$$\frac{\partial F_2(\omega, Q^2)}{\partial \log Q^2} \sim \bar{\alpha}_S(Q^2) \frac{\Phi(\omega, \bar{\gamma})}{\sqrt{-\bar{\gamma}^2 \tilde{K}'}} \exp \left\{ \int_{Q_0^2}^{Q^2} \frac{dq^2}{q^2} \bar{\gamma}(\bar{\alpha}_S(q^2)/\omega) \right\} \quad (47)$$

where

$$\tilde{K}' \equiv d\tilde{K}/d\gamma|_{\gamma=\bar{\gamma}} \quad \text{and} \quad \bar{\gamma} \equiv \bar{\gamma}(\bar{\alpha}_S(Q^2)/\omega).$$

This representation is applicable in the region  $\omega > \omega_L \equiv (4 \log 2) \bar{\alpha}_S(Q_0^2)$ . For smaller values of  $\omega$  the saddle-point estimate involves two (stationary phase) contributions which lead to a different representation of the integral. In other words (47) is not a valid approximation of the integral (34) for  $\omega < \omega_L$ . Unlike the case of fixed  $\alpha_S$ , the BFKL solution for running  $\alpha_S$  does not contain the branch point singularity at  $\omega = \omega_L$ , but rather it has (an infinite number of) poles in the  $\omega$  plane. The poles are determined by the starting point condition and hence are controlled by  $\bar{\alpha}_S$  at  $Q_0^2$ . It turns out that the leading pole singularity occurs at  $\omega < \omega_L(\alpha_S(Q_0^2))$  [14, 13].

In summary, we have confronted the collinear-factorization approach for the calculation of observable quantities at small  $x$  with the evaluation based on the  $k_T$ -factorization formula. For *fixed*  $\alpha_S$  both approaches are equivalent at the leading-twist level. In fact the insertion of the solution of the BFKL equation into the  $k_T$ -factorization formula provides a recipe for calculating  $\bar{\gamma} \equiv \gamma_{gg}$  and  $P_{gg}/\alpha_S$  as power series in  $\bar{\alpha}_S/\omega$ , where  $\omega$  is the moment index. The effect of introducing a *running*  $\alpha_S$  in the collinear-factorization formalism is summarized by (27) and (28). We noted that the leading singularity in the  $\omega$  plane is a branch point at  $\omega = \omega_L(Q_0^2)$  which is controlled by  $\alpha_S(Q_0^2)$  rather than  $\alpha_S(Q^2)$ , c.f. (31). That is the truly perturbative behaviour is hidden behind a non-perturbative contribution. In principle, the latter could be factored off and absorbed into the starting distribution. We then examined  $k_T$ -factorization with BFKL input with running  $\alpha_S$  and compared the predictions with those obtained from collinear factorization with running  $\alpha_S$ . We found the remarkable result that both the factorization prescriptions generate a perturbative expansion as a power series in  $\bar{\alpha}_S(Q^2)/\omega$  with exactly the *same* first few non-trivial terms, on top of the same DLL contribution. These terms are the most important perturbative contributions for the onset of the leading  $\log(1/x)$  behaviour in the HERA regime. In practice, in both the collinear- and  $k_T$ -factorization approaches, the leading singularity in the  $\omega$ -plane, which controls the small  $x$  behaviour, depends on  $Q_0^2$ . The location of the singularity is different, however. In the first case the power series in  $\bar{\alpha}_S(Q_0^2)/\omega$  builds up a branch point at  $\omega = \omega_L(Q_0^2)$ , whereas in the second case we generate a leading pole at a considerably smaller value of  $\omega$ . We conclude that the truly perturbative contributions in the two approaches are remarkably similar, but in practice they are partially hidden by non-perturbative terms.

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## Figure Caption

Fig. 1 Pictorial representation of the  $k_T$ -factorization formula of (1).

